Ginsparg-Wilson relation and the overlap formula

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The fermionic determinant of a lattice Dirac operator that obeys the Ginsparg-Wilson relation factorizes into two factors that are complex conjugates of each other. Each factor is naturally associated with a single chiral fermion and can be realized as an overlap of two many body vacua. [S0556-2821(98)10021-8]

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There has been a renewed interest in the Ginsparg-Wilson relation [1]

$$\gamma_5 D + D\gamma_5 = D\gamma_5 D. \tag{1}$$

One reason for this is an explicit example for D on the lattice [2,3]. The other reason is the argument that the perfect lattice action for the Dirac fermion implicitly satisfies the Ginsparg-Wilson relation [4].

A lattice Dirac operator satisfying the Ginsparg-Wilson relation is expected to have good chiral properties and recently it was shown that any lattice fermion action

$$S_F = \sum_x \bar{\psi} D \psi, \tag{2}$$

with D obeying the Ginsparg-Wilson relation (1), contains a continuous symmetry that can be viewed as lattice chiral symmetry [5]. This would imply that the fermionic determinant $\det D$ should factorize into two pieces with one being the complex conjugate of the other, if D obeys the Ginsparg-Wilson relation (1). Each piece would then be a single chiral determinant on the lattice associated with a left-handed or right-handed Weyl fermion. This is indeed the case for the particular form of D considered in [2,3] since the operator was obtained starting from the overlap formalism [6] where the factorization into chiral pieces is built in. Here I show that the factorization holds as long as D obeys the Ginsparg-Wilson relation. The two factors are complex conjugates of each other. Further each factor can be realized as an overlap of two many body vacua. The proof of factorization presented here simply amounts to retracing the steps in [3].

Following Refs. [2,3], we define an operator $\hat{H}$ through

$$D = 1 + \gamma_5 \hat{H}. \tag{3}$$

The Ginsparg-Wilson relation (1) reduces to

$$\hat{H}^2 = 1. \tag{4}$$

Therefore all the eigenvalues of $\hat{H}$ are ± 1. Starting in the chiral basis, let

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{5}$$

be the unitary matrix that diagonalizes $\hat{H}$ with

$$\hat{H}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \gamma & -\delta \end{pmatrix}. \tag{6}$$

Under a rotation by U,

$$U^\dagger D U = \begin{pmatrix} \alpha^\dagger & \gamma^\dagger \\ \beta^\dagger & \delta^\dagger \end{pmatrix}\begin{pmatrix} 2\alpha & 0 \\ 0 & 2\delta \end{pmatrix}. \tag{7}$$

If $\hat{H}$ has an equal number of positive and negative eigenvalues then α and δ are square matrices.[1] In this case,

$$\det D = \frac{\det \alpha^\dagger}{\det \delta} \det 2\alpha \det 2\delta = 2^{2N} \det \alpha \det \alpha^\dagger, \tag{8}$$

where α and δ are assumed to be $N\times N$ matrices. The first factor on the right side of the first equality in Eq. (8) follows from the fact that U is an unitary matrix. Eq. (8) is the factorization of D into two chiral factors one for each Weyl fermion. This factorization comes as no surprise since S_F in Eq. (2) contains lattice chiral symmetry [5].

Since the factorization was obtained by simply retracing the steps in [3], $\det \alpha$ should be associated with an overlap formula. To see this, consider the two many body Hamiltonians

$$\mathcal{H}^- = -a^\dagger \gamma_5 a; \quad \mathcal{H}^+ = -a^\dagger \hat{H} a, \tag{9}$$

with $a^\dagger$ and a are $2N$ fermion creation and annihilation operators obeying canonical anti-commutation relations. The matrix $\hat{H}$ is the $2N\times 2N$ matrix in Eq. (3) and γ_5 is trivially extended to be a $2N\times 2N$ matrix. Let $|0\pm\rangle$ be the many body ground states of $\mathcal{H}^\pm$. The identity in Appendix B of Ref. [7] implies that

$$\langle 0-|0+\rangle = \det \alpha, \tag{10}$$

with α being the submatrix of U in Eq. (5). Therefore each chiral factor in Eq. (8) is equal to an overlap of two many body vacua. Needless to say the phase of the many body

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[1]If $\hat{H}$ has an unequal number of positive and negative eigenvalues, α and δ are not square matrices and $\det D=0$ implying that the background gauge field has a nontrivial topology.

ground states plays a crucial role in the proper construction of chiral gauge theories [6] but it does not affect vector gauge theories since det D is real and positive and independent of the phase of det α.

A specific form for $\hat{H}$ in [2,3] is arrived at by writing $\hat{H}=H/\sqrt{H^2}$ with H being the Wilson realization of the continuum $\gamma_5[\gamma_\mu(\partial_\mu+iA_\mu(x)-m)]$ operator on the lattice for some $m>0$. Any other discretization of the continuum operator can also be used. The perfect action implicitly defined in [4] will result in a different $\hat{H}$ on the lattice. This one would be as good as the one in [3] in the continuum limit. It can also be inserted in Eq. (9) and the determinant can be thought of as an overlap of two vacua. Equation (3) can be thought of as a method to construct a Dirac operator on the lattice with a lattice chiral symmetry by starting from some discretization of the continuum Dirac operator on the lattice that does not posses any chiral symmetry.

We have shown that the determinant of a lattice Dirac operator that obeys the Ginsparg-Wilson relation factorizes into pieces. The two pieces are complex conjugates of each other. Each piece is the determinant of a Weyl fermion and can be thought of as as an overlap of two many body vacua. It should be emphasized again that there are potentially many choices for $\hat{H}$ in Eq. (3) and all these are expected to describe the same theory in the continuum limit.

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